FURTHER MATHEMATICS

Paper 9231/11
Further Pure Mathematics 1

Key messages

Candidates should show all the steps in their solutions, particularly when proving a given result.

Candidates should read questions carefully so that they answer all aspects in adequate depth. They should take note of where exact answers are required.

Any sketch graphs should be fully labelled and carefully drawn to show behaviour at limits and at significant points.

Candidates need to be familiar with which formulae are included in the list of formulae (MF19).

General comments

The majority of candidates demonstrated good knowledge across the whole syllabus. They showed their working clearly and were generally accurate in their handling of algebra. It seemed that almost all candidates were able to complete the paper in the time allowed.

Comments on specific questions

Question 1

(a) There were many very good solutions seen for this question. Candidates found the partial fractions efficiently and showed sufficient correct terms to make the cancellation clear. The best solutions wrote the terms below each other so the pattern was clearly visible. Those candidates who had difficulty in finding the partial fractions often tried to cancel terms of different form.

(b) Those who had a correct answer for **part (a)** had no problems dealing with the sum to infinity. Most of them remembered to discard the negative value because a is positive.

Question 2

This question was generally well done. In all vector questions candidates should use correct notation, distinguishing between position vectors and direction vectors. It is also advisable to check arithmetic carefully as a small error can change the nature of a whole question.

(a) The method for this was well understood. The most frequent errors were numerical, seen in finding the direction vectors and in taking the cross-product.

(b) Most candidates were able to find the perpendicular distance and remembered to divide by the length of the normal vector.

(c) Most candidates could write down the coordinates or position vector of point D in terms of a parameter. Having found this parameter, a number of them gave the position vector of the point rather than its coordinates as specified in the question.

Question 3

When working with inequalities, candidates should ensure that the steps performed maintain the direction of the inequality sign.

(a) There were some excellent, efficient and well-argued solutions for this question and most candidates clearly understood the steps necessary for a proof by induction. They needed to start by deciding that the hypotheses were, in fact, $u_1 > 4$ (given) and $u_k > 4$ in order to proceed. A surprising number used the recurrence relation to find u_2 in an attempt to justify the base case. When dealing with an inequality it is often easier to deal with the sign of an expression and a good number considered $u_k - 4$. When this was expressed as $\frac{(u_k - 3)(u_k - 4)}{2u_k}$ a correct argument usually followed. Those who turned the expression into a sum of fractions found it difficult to produce an argument using inequalities.

(b) This was often answered better than **part (a)**. Candidates found $u_n - u_{n+1}$ and showed it to be positive by a correct argument. Many candidates wrote it down as positive and showed that it satisfied the inequality, but they omitted to state that $u_n > 0$ before multiplying through.

Question 4

The method needed here was clearly well understood by many candidates. The most common errors were numerical ones.

(a) The substitution $x = y^{-\frac{1}{3}}$ was seen most frequently and candidates attempted cubing correctly.

(b) The relationship between the coefficients in **part (a)** and the required sum was usually written down and used correctly.

(c) Most candidates appreciated the relationship between the three parts of the question and used the recurrence method correctly to find the required sum. A few of them tried to return to some version of the original equation without success.

Question 5

(a) The majority of candidates either considered the discriminant of $x^2 + x + 1 = 0$ or solved it to produce complex roots in order to explain why there are no vertical asymptotes. The horizontal asymptote was almost always correct.

(b) The straightforward differentiation method to find stationary points was almost always successful, the only errors being numerical.

Those who tried the alternative method of using the discriminant to find extreme values of *y* usually found the points (0, –1) and (–2, 3) but did not fully justify why they were stationary points.

(c) There were some very well drawn graphs, with two clear approaches to the horizontal asymptote and the stationary points shown correctly. Many candidates seemed to believe that a graph can never cross an asymptote. This resulted in many graphs with more than one *y* value for each *x* value. The coordinates of intersections with the axes were almost always correct and usually clearly stated.

(d) The reflection in the *x*-axis was done correctly. The inequality produced some incorrect use of –1, 2 and 3, suggesting that candidates were unsure about the relationship between the question and the graph they had drawn.

Question 6

(a) Most candidates used the correct domain and produced a roughly correct curve. The only part to cause difficulty was the gradient at the extremities.

(b) Candidates should remember that an integral sign needs the $\mathrm{d}x$ or in this case $\mathrm{d}\theta$ to state what integration is to be performed (this can be particularly helpful when performing a substitution). For this question integration by parts was required. The differentiation of $\tan^{-1}\left(\frac{1}{2}\theta\right)$ often involved a lost $\frac{1}{2}$ at some point.

Some candidates simply omitted this part of the question or tried to express $\tan^{-1}\left(\frac{1}{2}\theta\right)$ in different form. $\dfrac{\sin^{-1}\left(\frac{1}{2}\theta\right)}{\cos^{-1}\left(\frac{1}{2}\theta\right)}$ or $\dfrac{\cos\left(\frac{1}{2}\theta\right)}{\sin\left(\frac{1}{2}\theta\right)}$ were not uncommon.

(c) There has been an improvement in the number of candidates who realised that they were being asked to consider the maximum value of $r\cos\theta$ rather than some other distance. The differentiation of $\sqrt{\tan^{-1}\left(\frac{1}{2}\theta\right)}$ needed clear organisation, and again there were missing $\frac{1}{2}$s.

A particularly elegant solution was to differentiate $x^2 = r^2\cos^2\theta = \tan^{-1}\left(\frac{1}{2}\theta\right)\cos^2\theta$ which produced some very good proofs of the given result.

Most candidates demonstrated the required change of sign to show the position of a root. Correct numerical values were needed.

Question 7

(a) Most candidates correctly evaluated the determinant, with few numerical or sign errors. It was quite common to see the fact that A is non-singular interpreted as $\det \mathrm{A} > 0$ rather than $\det \mathrm{A} \neq 0$. Those who understood the meaning of non-singular often wrote '$k > 5$ or $k < 5$' as an acceptable alternative to $k \neq 5$.

(b) There were few completely correct answers for this question. Those using row operations quickly ran into problems with algebra and did not reach the end. Those using cofactors made sign errors or forgot to divide by the determinant. A quick solution was to pre-multiply by a matrix of unknowns and solve three simultaneous equations.

(c) Candidates usually found the matrix **BA** correctly. The majority of them then made it clear that they were looking for a matrix with three rows and two columns to be compatible with the equation.

BAC $= \begin{pmatrix} 2 & 1 \\ k & 4 \end{pmatrix}$. The question asked for an example, but there were many attempts to find a unique solution. With 4 equations and 6 unknowns this was not possible. There were also many attempts to use the inverse of a non-square matrix. A significant number of candidates went back to the question and spotted a simple solution.

(d) There has been a marked improvement in candidates' work on this topic and this question was well done. Very few candidates tried to use the incorrect $y = mx + c$ rather than $y = mx$. The majority of them made it clear that they were looking for invariant lines rather than invariant points and wrote down a correct quadratic equation in the gradient. They then used the discriminant correctly to find the range of possible values of *k*, though $k \neq 8$ was invariably omitted. Errors were usually numerical.

FURTHER MATHEMATICS

Paper 9231/12
Further Pure Mathematics 1

Key messages

Candidates should show all the steps in their solutions, particularly when proving a given result.

Candidates should read questions carefully so that they answer all aspects in adequate depth. They should take note of where exact answers are required.

Any sketch graphs should be fully labelled and carefully drawn to show behaviour at limits and at significant points.

Candidates need to be familiar with which formulae are included in the list of formulae (MF19).

General comments

The majority of candidates demonstrated good knowledge across the whole syllabus. They showed their working clearly and were generally accurate in their handling of algebra. It seemed that almost all candidates were able to complete the paper in the time allowed.

Comments on specific questions

Question 1

(a) There were many very good solutions seen for this question. Candidates found the partial fractions efficiently and showed sufficient correct terms to make the cancellation clear. The best solutions wrote the terms below each other so the pattern was clearly visible. Those candidates who had difficulty in finding the partial fractions often tried to cancel terms of different form.

(b) Those who had a correct answer for **part (a)** had no problems dealing with the sum to infinity. Most of them remembered to discard the negative value because a is positive.

Question 2

This question was generally well done. In all vector questions candidates should use correct notation, distinguishing between position vectors and direction vectors. It is also advisable to check arithmetic carefully as a small error can change the nature of a whole question.

(a) The method for this was well understood. The most frequent errors were numerical, seen in finding the direction vectors and in taking the cross-product.

(b) Most candidates were able to find the perpendicular distance and remembered to divide by the length of the normal vector.

(c) Most candidates could write down the coordinates or position vector of point D in terms of a parameter. Having found this parameter, a number of them gave the position vector of the point rather than its coordinates as specified in the question.

Question 3

When working with inequalities, candidates should ensure that the steps performed maintain the direction of the inequality sign.

(a) There were some excellent, efficient and well-argued solutions for this question and most candidates clearly understood the steps necessary for a proof by induction. They needed to start by deciding that the hypotheses were, in fact, $u_1 > 4$ (given) and $u_k > 4$ in order to proceed. A surprising number used the recurrence relation to find u_2 in an attempt to justify the base case. When dealing with an inequality it is often easier to deal with the sign of an expression and a good number considered $u_k - 4$. When this was expressed as $\frac{(u_k - 3)(u_k - 4)}{2u_k}$ a correct argument usually followed. Those who turned the expression into a sum of fractions found it difficult to produce an argument using inequalities.

(b) This was often answered better than **part (a)**. Candidates found $u_n - u_{n+1}$ and showed it to be positive by a correct argument. Many candidates wrote it down as positive and showed that it satisfied the inequality, but they omitted to state that $u_n > 0$ before multiplying through.

Question 4

The method needed here was clearly well understood by many candidates. The most common errors were numerical ones.

(a) The substitution $x = y^{-\frac{1}{3}}$ was seen most frequently and candidates attempted cubing correctly.

(b) The relationship between the coefficients in **part (a)** and the required sum was usually written down and used correctly.

(c) Most candidates appreciated the relationship between the three parts of the question and used the recurrence method correctly to find the required sum. A few of them tried to return to some version of the original equation without success.

Question 5

(a) The majority of candidates either considered the discriminant of $x^2 + x + 1 = 0$ or solved it to produce complex roots in order to explain why there are no vertical asymptotes. The horizontal asymptote was almost always correct.

(b) The straightforward differentiation method to find stationary points was almost always successful, the only errors being numerical.

Those who tried the alternative method of using the discriminant to find extreme values of *y* usually found the points (0, –1) and (–2, 3) but did not fully justify why they were stationary points.

(c) There were some very well drawn graphs, with two clear approaches to the horizontal asymptote and the stationary points shown correctly. Many candidates seemed to believe that a graph can never cross an asymptote. This resulted in many graphs with more than one *y* value for each *x* value. The coordinates of intersections with the axes were almost always correct and usually clearly stated.

(d) The reflection in the *x*-axis was done correctly. The inequality produced some incorrect use of –1, 2 and 3, suggesting that candidates were unsure about the relationship between the question and the graph they had drawn.

Question 6

(a) Most candidates used the correct domain and produced a roughly correct curve. The only part to cause difficulty was the gradient at the extremities.

(b) Candidates should remember that an integral sign needs the $\mathrm{d}x$ or in this case $\mathrm{d}\theta$ to state what integration is to be performed (this can be particularly helpful when performing a substitution). For this question integration by parts was required. The differentiation of $\tan^{-1}\left(\frac{1}{2}\theta\right)$ often involved a lost $\frac{1}{2}$ at some point.

Some candidates simply omitted this part of the question or tried to express $\tan^{-1}\left(\frac{1}{2}\theta\right)$ in different form. $\dfrac{\sin^{-1}\left(\frac{1}{2}\theta\right)}{\cos^{-1}\left(\frac{1}{2}\theta\right)}$ or $\dfrac{\cos\left(\frac{1}{2}\theta\right)}{\sin\left(\frac{1}{2}\theta\right)}$ were not uncommon.

(c) There has been an improvement in the number of candidates who realised that they were being asked to consider the maximum value of $r\cos\theta$ rather than some other distance. The differentiation of $\sqrt{\tan^{-1}\left(\frac{1}{2}\theta\right)}$ needed clear organisation, and again there were missing $\frac{1}{2}$s.

A particularly elegant solution was to differentiate $x^2 = r^2\cos^2\theta = \tan^{-1}\left(\frac{1}{2}\theta\right)\cos^2\theta$ which produced some very good proofs of the given result.

Most candidates demonstrated the required change of sign to show the position of a root. Correct numerical values were needed.

Question 7

(a) Most candidates correctly evaluated the determinant, with few numerical or sign errors. It was quite common to see the fact that A is non-singular interpreted as $\det A > 0$ rather than $\det A \neq 0$. Those who understood the meaning of non-singular often wrote '$k > 5$ or $k < 5$' as an acceptable alternative to $k \neq 5$.

(b) There were few completely correct answers for this question. Those using row operations quickly ran into problems with algebra and did not reach the end. Those using cofactors made sign errors or forgot to divide by the determinant. A quick solution was to pre-multiply by a matrix of unknowns and solve three simultaneous equations.

(c) Candidates usually found the matrix **BA** correctly. The majority of them then made it clear that they were looking for a matrix with three rows and two columns to be compatible with the equation.

BAC $= \begin{pmatrix} 2 & 1 \\ k & 4 \end{pmatrix}$. The question asked for an example, but there were many attempts to find a unique solution. With 4 equations and 6 unknowns this was not possible. There were also many attempts to use the inverse of a non-square matrix. A significant number of candidates went back to the question and spotted a simple solution.

(d) There has been a marked improvement in candidates' work on this topic and this question was well done. Very few candidates tried to use the incorrect $y = mx + c$ rather than $y = mx$. The majority of them made it clear that they were looking for invariant lines rather than invariant points and wrote down a correct quadratic equation in the gradient. They then used the discriminant correctly to find the range of possible values of *k*, though $k \neq 8$ was invariably omitted. Errors were usually numerical.

FURTHER MATHEMATICS

Paper 9231/13
Further Pure Mathematics 1

Key messages

Candidates should show all the steps in their solutions, particularly when proving a given result.

Candidates should read questions carefully so that they answer all aspects in adequate depth. They should take note of where exact answers are required.

Any sketch graphs should be fully labelled and carefully drawn to show behaviour at limits and at significant points.

Candidates need to be familiar with which formulae are included in the list of formulae (MF19).

General comments

The majority of candidates demonstrated good knowledge across the whole syllabus. They showed their working clearly and were accurate in their handling of algebra. It seemed that almost all candidates were able to complete the paper in the time allowed.

Comments on specific questions

Question 1

(a) The majority of candidates sketched the curve correctly.

(b) Many candidates understood that they needed to use the graph in **part (a)** and reflect in the *y*-axis the part corresponding to positive values of *x*. There should have been a cusp at $x = 0$ but this was frequently shown as a turning point. The critical value of $x = \frac{1}{3}$ was usually correct.

The most common error was to reflect the part of the graph for $x < 0$ on to the graph for $x > 0$.

Question 2

(a) This topic seemed to be well understood and almost all candidates produced fully correct solutions.

(b) The determinant was usually correct. The substitution of $\alpha\beta\gamma = 2$ needed to be shown clearly as the final answer of 0 was given in the question.

Question 3

(a) This was generally well done, with only a few candidates finding an incorrect oblique asymptote.

(b) Most candidates considered the discriminant, usually saying determinant < 0 for no possible values of *y*. They then needed to justify why this led to the given range of values. The most usual ways to do so were factorising or providing a small sketch.

Those who chose to consider stationary values usually got no further than finding (0, 1) and (2, 4a + 1). The best solutions then showed that these points are a local maximum and a local minimum respectively. A comment that the graph has two branches or does not cross $x = 1$ was needed to complete the argument.

Candidates who tried to reason from the equation often multiplied through by $(x-1)$. The proof needed to be split into the two cases, $x < 1$ and $x > 1$, for work with inequalities to be correct.

(c) Candidates should label their asymptotes clearly, and take care that their graph approaches them correctly. The shape and positioning of the two branches was usually correct.

Question 4

(a) There were many clearly presented and fully correct solutions using the terms in the form $e^{(r+2)x} - 2e^{(r+1)x} + e^{rx}$. The three parts meant that a block of three consecutive terms was needed to show the pattern of cancelling. This was not always done, leading to errors in the final answer. Other arrangements were also used successfully.

Several candidates noticed that it was a geometric series with first term $e^x(e^{2x} - 2e^x + 1)$ and common ratio e^x, they used this fact successfully to find the sum.

(b) This proved challenging for many candidates. Most candidates realised that something must happen to the one or two terms containing e^{nx} and so simply ignored them. A recognition that $e^{nx} \to 0$ was required. Those candidates who had noticed the geometric series realised that this was the case for a sum to infinity to exist.

(c) Many candidates did not use the laws of logarithms correctly.

Question 5

(a) This was almost always correct.

(b) The structure of proof by induction is clearly well known, but there needs to be precision in the wording. In a question like this, the base case and inductive hypothesis should each contain an algebraic statement. The base case should be said to be true and the hypothesis to be assumed for a particular value of *k* and not for all *k*. The step from *k* to *k* + 1 was usually done well, with the intermediate step shown. Most candidates remembered to say 'for all positive integers *n*' in their final conclusion.

(c) There were many fully correct solutions. Most candidates were clearly looking for invariant lines rather than points and wrote down a correct equation. The most efficient way of solving this was to write it as $\frac{1}{a}(1+m) = m(b+n)(1+m)$. Use of the common factor 1 + *m* then gave a quick route to the answer, as long as it was not cancelled without consideration. Many candidates solved the quadratic equation using the formula which gave more chance of error.

Question 6

(a) This part was usually answered correctly.

(b) A polar graph should have the pole and initial line clearly marked. Several candidates used cartesian axes instead which was incorrect. Most candidates realised that *r* is strictly decreasing but the slope at the extreme points needed careful consideration.

(c) A good proportion of candidates were able to carry out the substitution as requested. They understood that all parts of the integral must be considered: $\tan\theta$ to t, $d\theta$ to dt and $\left(0, \frac{\pi}{4}\right)$ to $(0, 1)$.

Stronger answers used completing the square and the standard $\tan^{-1}$ integral. Many candidates attempted to use a ln integration even though the derivative of the denominator was not present.

Question 7

This topic is clearly well understood. Candidates are advised to distinguish carefully between coordinates, position vectors and direction vectors and to use correct notation. Most errors were numerical, particularly in calculating the cross product.

(a) This part was generally very well done, and most candidates understood that they needed to show all of their working because they were proving a given answer.

(b) (i) This was usually successful. The most common error was to omit **r** = from the equation.

(ii) This was also well done apart from numerical errors.

(c) Despite numerical errors from previous parts it was clear that candidates knew what was required. Very few gave an angle other than the acute one as their final answer.

FURTHER MATHEMATICS

Paper 9231/21
Further Pure Mathematics 2

Key messages

- Candidates should show all the steps in their solutions, particularly when proving a given result.
- Candidates should read questions carefully so that they answer all aspects in adequate depth, noting when an answer is required in a certain form or in terms of a given variable.
- Candidates should make use of results derived or given in earlier parts of a question or given in the list of formulae (MF19).

General comments

The majority of candidates demonstrated very good knowledge across the whole syllabus and showed their working clearly. In their handling of algebra and calculus, candidates were accurate, and they also displayed understanding of linear algebra. It seemed that all candidates were able to complete the paper in the time allowed.

Sometimes candidates did not fully justify their answers. This was particularly important where answers were given within the question and was relevant to **Questions 2(a)**, **4(a)**, **5(a)**, **7(a)** and **8(c)**.

Comments on specific questions

Question 1

The best solutions recalled the correct formula for the length of a polar curve. They answered the question fully by rearranging to find α in terms of *s*, maintaining accuracy throughout.

Question 2

(a) This part of the question was done well. The majority of candidates substituted sinh in terms of exponentials accurately and wrote out the full expansion of $\sinh^2 x$ to justify the given identity.

(b) The majority of candidates applied the identity given in **part (a)** and set the discriminant as positive to derive the correct set of values of *k*.

Question 3

(a) Almost all candidates approached this question correctly and gave complete solutions to a high standard. Some errors were seen in solving linear equations to find the particular integral. Other errors involved notation. A few candidates gave their answer in the form of an expression instead of an equation.

(b) Most candidates successfully stated an approximate value using their answer to **part (a)**.

Question 4

(a) Most candidates formed a correct expression for the sum of the areas of the rectangles. Better responses identified this correctly as the sum of a geometric progression correctly and applied the formula. They stated the first term and common ratio and proceeded to derive the given upper bound.

(b) The best solutions adapted the method used in **part (a)** to derive a suitable lower bound correctly.

(c) Some candidates were able to simplify their $U_N - L_N$ to N^{-1} and deduce the required least value of N.

Question 5

(a) The majority of candidates differentiated both sides of the equation accurately using implicit differentiation, showing sufficient working to justify the given answer.

(b) Better responses used implicit differentiation accurately again to find an equation involving the second derivative. Most successful attempts retained the factorisation in their expression and maintained accuracy when substituting to find $f''(0)$. They substituted $f(0)=0$, $f'(0)=-\frac{3}{4}$ and $f''(0)=\frac{9}{32}$ into the Maclaurin's series for y and reached a fully correct solution.

Question 6

The best responses differentiated vx using the product rule and, following substitution, eliminated y from the given equation. After separating variables, they integrated the left-hand side to $\sinh^{-1}$ using the relevant result from the list of formulae (MF 19), then maintained accuracy when substituting in the initial conditions. By converting $\sinh^{-1}$ to logarithmic form and squaring to eliminate the radical, they were able to reach a correct expression for y as a polynomial in x.

Question 7

(a) Most candidates used de Moivre's theorem successfully to find the required $\operatorname{cosec} 7\theta$. A few candidates had difficulty in translating their expression in cos and sin into an expression in cosec. There were some elegant uses of $\cot^2\theta = \operatorname{cosec}^2\theta - 1$ and the best responses showed full details of all the steps in the working. This was particularly important since the answer was given in the question.

(b) Better responses made the connection with the equation given in **part (a)**, justifying the relationship, then used $\operatorname{cosec} 7\theta = 2$ to find all seven distinct solutions.

Question 8

(a) & (b) These parts of the question were done well, though a few candidates accepted zero eigenvectors without checking for errors in their working. Some candidates spent time finding $\det(\mathbf{A}-\lambda\mathbf{I})$ instead of reading directly from the diagonal of the matrix.

(b) Better solutions displayed accuracy throughout, both when substituting into the characteristic equation and when making $\mathbf{A}+6\mathbf{I}$ the subject before squaring both sides.

FURTHER MATHEMATICS

Paper 9231/22
Further Pure Mathematics 2

Key messages

- Candidates should show all the steps in their solutions, particularly when proving a given result.
- Candidates should read questions carefully so that they answer all aspects in adequate depth, noting when an answer is required in a certain form or in terms of a given variable.
- Candidates should make use of results derived or given in earlier parts of a question or given in the list of formulae (MF19).

General comments

The majority of candidates demonstrated very good knowledge across the whole syllabus and showed their working clearly. In their handling of algebra and calculus, candidates were accurate, and they also displayed understanding of linear algebra. It seemed that all candidates were able to complete the paper in the time allowed.

Sometimes candidates did not fully justify their answers. This was particularly important where answers were given within the question and was relevant to **Questions 2(a)**, **4(a)**, **5(a)**, **7(a)** and **8(c)**.

Comments on specific questions

Question 1

The best solutions recalled the correct formula for the length of a polar curve. They answered the question fully by rearranging to find α in terms of s, maintaining accuracy throughout.

Question 2

(a) This part of the question was done well. The majority of candidates substituted sinh in terms of exponentials accurately and wrote out the full expansion of $\sinh^2 x$ to justify the given identity.

(b) The majority of candidates applied the identity given in **part (a)** and set the discriminant as positive to derive the correct set of values of k.

Question 3

(a) Almost all candidates approached this question correctly and gave complete solutions to a high standard. Some errors were seen in solving linear equations to find the particular integral. Other errors involved notation. A few candidates gave their answer in the form of an expression instead of an equation.

(b) Most candidates successfully stated an approximate value using their answer to **part (a)**.

Question 4

(a) Most candidates formed a correct expression for the sum of the areas of the rectangles. Better responses identified this correctly as the sum of a geometric progression correctly and applied the formula. They stated the first term and common ratio and proceeded to derive the given upper bound.

(b) The best solutions adapted the method used in **part (a)** to derive a suitable lower bound correctly.

(c) Some candidates were able to simplify their $U_N - L_N$ to N^{-1} and deduce the required least value of *N*.

Question 5

(a) The majority of candidates differentiated both sides of the equation accurately using implicit differentiation, showing sufficient working to justify the given answer.

(b) Better responses used implicit differentiation accurately again to find an equation involving the second derivative. Most successful attempts retained the factorisation in their expression and maintained accuracy when substituting to find $f''(0)$. They substituted $f(0) = 0$, $f'(0) = -\frac{3}{4}$ and $f''(0) = \frac{9}{32}$ into the Maclaurin's series for *y* and reached a fully correct solution.

Question 6

The best responses differentiated vx using the product rule and, following substitution, eliminated *y* from the given equation. After separating variables, they integrated the left-hand side to $\sinh^{-1}$ using the relevant result from the list of formulae (MF 19), then maintained accuracy when substituting in the initial conditions. By converting $\sinh^{-1}$ to logarithmic form and squaring to eliminate the radical, they were able to reach a correct expression for *y* as a polynomial in x.

Question 7

(a) Most candidates used de Moivre's theorem successfully to find the required $\operatorname{cosec}7\theta$. A few candidates had difficulty in translating their expression in cos and sin into an expression in cosec. There were some elegant uses of $\cot^2\theta = \operatorname{cosec}^2\theta - 1$ and the best responses showed full details of all the steps in the working. This was particularly important since the answer was given in the question.

(b) Better responses made the connection with the equation given in **part (a)**, justifying the relationship, then used $\operatorname{cosec}7\theta = 2$ to find all seven distinct solutions.

Question 8

(a) & **(b)** These parts of the question were done well, though a few candidates accepted zero eigenvectors without checking for errors in their working. Some candidates spent time finding $\det(\mathbf{A} - \lambda\mathbf{I})$ instead of reading directly from the diagonal of the matrix.

(b) Better solutions displayed accuracy throughout, both when substituting into the characteristic equation and when making $\mathbf{A} + 6\mathbf{I}$ the subject before squaring both sides.

FURTHER MATHEMATICS

Paper 9231/23
Further Pure Mathematics 2

Key messages

- Candidates should show all the steps in their solutions, particularly when proving a given result.
- Candidates should read questions carefully so that they answer all aspects in adequate depth, noting when an answer is required in a certain form or in terms of a given variable.
- Candidates should make use of results derived in earlier parts of a question or given in the list of formulae (MF19).

General comments

There were many responses of a very high standard. The majority of candidates demonstrated very good knowledge across the whole syllabus and showed their working clearly. In their handling of algebra and calculus, they were accurate, and they also displayed understanding of linear algebra. It seemed that all candidates were able to complete the paper in the time allowed.

Sometimes candidates did not justify their answers fully. This is particularly important where answers are given within the question.

Comments on specific questions

Question 1

The best solutions showed clear working, starting from z^3 in exponential form with the correct argument, and listed all three roots in the required form.

Question 2

(a) This part was done well, with most candidates stating the correct derivatives and dividing the second derivative by 2 to obtain the coefficient of x^2.

(b) The majority of candidates used the formula for the integral of sec*x* given in the list of formulae (MF19). A few candidates derived an incorrect formula or used a calculator without showing working.

Question 3

(a) This part of the question was well done with better responses showing full working when factorising $\det(\mathbf{A} - \lambda\mathbf{I})$.

(b) Better solutions maintained accuracy and used correct notation when manipulating the characteristic equation. Some candidates substituted in for $\mathbf{A}^2$ and $\mathbf{I}$, which was not required to answer the question.

Question 4

(a) The majority of candidates found the first derivative correctly using parametric differentiation.

(b) Attempts to find the second derivative varied in length. The best solutions showed the required level of algebraic fluency and did not omit to divide by $\frac{dx}{dt}$ after differentiating with respect to t.

Question 5

The majority of candidates divided both sides of the equation by $x^2 + 7x$ and found the integrating factor correctly by using partial fractions or completing the square. After multiplying both sides of the equation by $x(x+7)^{-1}$, the best answers applied $x = x + 7 - 7$ or integration by parts in order to integrate the right-hand side. They also maintained accuracy when substituting in the initial conditions.

Question 6

(a) Better solutions started by forming a correct expression for the sum of the areas of the rectangles. They proceeded to apply laws of logarithms clearly and finished by fully justifying the given answer.

(b) Some candidates spotted that there were $n-1$ terms and so worked with n^{n-1}, which led to a correct lower bound. A few candidates spotted that the difference between U_n and L_n is $\frac{1}{n}\ln 2$, which they used to reach a correct solution for this part of the question and the next part.

(c) The best solutions showed enough working to justify that the difference between U_n and L_n is proportional to $\frac{1}{n}$, hence justifying the given limit.

Question 7

(a) Almost all candidates knew how to approach this question and completed it to a high standard. The majority of them took the most efficient approach of substituting $y = k$ to find the particular integral. There was some inaccuracy when solving linear equations to find the values of the constants and some problems with notation were seen. A few candidates gave expressions instead of equations as their answer.

(b) Those who were successful with **part (a)** usually demonstrated the skills to derive and solve an equation of the form $\sinh ax = b$, and so were successful with this part also.

Question 8

(a) Almost all candidates recognised the integral as $-\frac{\cos^{n+1}\theta}{n+1} + C$, although it was common to see the arbitrary constant omitted.

(b) The best solutions carried out integration by parts first, using the result from **part (a)** to inform their choice of parts. They then replaced $\cos^2\theta$ with $1-\sin^2\theta$ to derive the given reduction formula successfully. The few candidates who applied $\cos^2\theta = 1-\sin^2\theta$ first, before integrating by parts, or who split the product as $\sin^{m-1}\theta\cos^n\theta\sin\theta$ were usually successful also.

(c) After expanding $\left(z+z^{-1}\right)^5$ using the binomial expansion, better responses grouped together terms clearly before applying the identity $z^n + z^{-n} = 2\cos n\theta$ to fully justify their answer.

(d) This part was well done with the majority of candidates applying the reduction formula accurately.

FURTHER MATHEMATICS

Paper 9231/31
Further Mechanics

Key messages

A diagram is often an invaluable tool in helping a candidate to make good progress. This is particularly the case when forces or velocities are involved. If a diagram is given on the question paper it may be sufficient to annotate that diagram, although candidates are always free to draw their own diagrams as well.

When a result is given in a question, candidates must take care to give sufficient detail in their working, so that their solution is clear and complete. In all questions, however, candidates are advised to show all their working, as method is important as well as accuracy.

General comments

Candidates are encouraged to draw a suitable diagram as this helps them to understand the problem and model it correctly. For example, in **Question 3**, candidates who drew a clear diagram realised that the acceleration of the particle had a negative sign and were able to write in the correct form the differential equation describing its motion.

Candidates should pay particular attention to the mathematical vocabulary used in the text of the question and also to what the question asks them to do. For example, **Question 4** used the word *shell* three times and mentioned a *hollow cylinder* twice, and **Question 6(a)** asked for the speed of sphere A in addition to that of sphere B. Candidates who understood the text correctly managed to set up the correct moments equation in **Question 4** and to obtain a fully correct solution in **Question 6(a)**.

Candidates are reminded that, when the answer is given, they are expected to show their working in full, even if it involves the use of simple algebra. This was relevant to **Question 6(a)** and **Question 7(a)**.

Comments on specific questions

Question 1

(a) Most candidates had no difficulty in stating the values of the components of the tension in the string and then went on to answer the question correctly. Better responses used Pythagorean triples, whereas weaker responses determined the magnitude of the angle θ, as an intermediate step, even though this was not needed. A common mistake was to consider the vertical component of the tension as 10 g, i.e.,100 N, rather than 10 N as stated in the question.

(b) This part of the question was also well answered by many candidates. They needed to apply Hooke's law correctly to determine the extension of the string and then obtain the answer using trigonometry. A common mistake was to confuse sine and cosine thus obtaining an answer of 0.6, or to omit to calculate the extension.

Question 2

This question proved challenging for some candidates, who had difficulty in writing the correct equation for the conservation of energy, often making errors in the signs of the different terms. Some candidates worked out the value of the tension of the string at point A, even though this was not necessary. A common error was to state that the velocity of the particle is zero when the string goes slack, and to use that value in the equation for the conservation of energy. In the best responses, candidates drew very clear diagrams and used them effectively to obtain the correct answer, often in an elegant and efficient way.

Question 3

Many candidates were able to set up a differential equation and to solve it, including applying the boundary conditions. However, only some candidates realised that acceleration and velocity had opposite directions and therefore the correct differential equation was $\frac{\mathrm{d}v}{\mathrm{d}t} = -\frac{4000}{(5t+4)^3}$. Candidates who drew a diagram representing the situation and included the directions of acceleration and velocity were able to write the correct differential equation. They usually went on to provide a correct solution.

Question 4

(a) This part of the question was answered correctly by few candidates. Most candidates wrote the equation of a hemisphere on top of a solid cylinder, even though the question explicitly mentioned a hemispherical shell and a hollow cylinder. Weaker responses included a moments equation where the terms were not consistent dimensionally. This is an easy check to perform, and candidates are encouraged to use it. Without a dimensionally consistent equation it is not possible to proceed further.

(b) In better responses, candidates drew a correct diagram to answer this part question. With the aid of the diagram, they were able to obtain the inequality linking $\tan\theta$, a, h and $\bar{x}$ (the distance of the centre of the mass obtained in the previous part) and to solve it. Common errors involved writing an equation instead of an inequality, writing the incorrect direction of the inequality, or comparing $\tan\theta$ with $\frac{\bar{x}}{a}$ and not with $\frac{a}{\bar{x}}$.

Question 5

To obtain the correct answer for this question, candidates were required to perform three steps:

- Apply Newton's Second Law horizontally at point *A*. They needed to use trigonometry to determine the expression for the radius and simplify the equation to determine the tension on the string at point *A*: $T = 3am\omega^2$.
- Apply Newton's Second Law and proceed to find the tension at point *B*: $T = \frac{3}{4}amk^2\omega^2$.
- Equate the expressions for the tensions and solve the resulting equation in k to give the solution $k = 2$.

Some excellent responses were seen as the question gave candidates the opportunity to respond in an elegant and efficient way.

Some candidates performed the first two steps very quickly, but then did not realise that the tension at point *A* was equal to that at point *B* as those were the endpoint of the string. Consequently, they rarely managed to obtain the correct answer.

Other candidates found this question challenging in various ways. Some did not apply Newton's Second Law correctly or could not determine the correct expression for the radii of the two circles. A few candidates did not use the correct value for the mass of the particle attached at point *B*.

Question 6

(a) Many candidates showed a great degree of confidence in writing and solving successfully the system of equations obtained from the conservation of linear momentum and Newton's experimental law to obtain the given answer. However, some of them omitted to determine the expression for the speed of sphere A along the line of centres, even though this was explicitly requested in the question. Those candidates did not produce a fully correct solution in this part although, in some cases, they worked it out subsequently in **part (b)**. Even though it was not required in **part (a)**, some candidates worked out the expression for the speed of sphere A, including the component perpendicular to the line of centres. They later used it in **part (b)**, usually with good success. A common mistake was not to include the mass in the equation for the conservation of momentum, i.e., writing $v + kw = u\cos\alpha$ instead of $mv + kmw = mu\cos\alpha$. As a result, these candidates did not provide a complete argument leading to the given answer as required.

(b) This part question proved more challenging than **part (a)**. Many candidates had difficulty in determining the kinetic energy of sphere *A* after the collision. The most common mistake was to omit the component of the speed perpendicular to the line of centres $(u\sin\alpha)$. Another common mistake was to write the mass of sphere *B* as *m*, rather than *km*. Those who wrote a correct equation for the kinetic energy of the spheres usually had no difficulty in solving the equation to obtain the correct answer using competent algebraic manipulation.

Question 7

(a) This part question required candidates to obtain a given equation by working out expressions for the <u>horizontal</u> displacements of particles P and Q. Candidates who wrote the correct displacements had no problem in obtaining the required answer, often in a convincing and elegant manner. Weaker responses considered the <u>vertical</u> displacements of the particles instead which was incorrect. Another common mistake was to use time $T+1$ for particle Q and T for particle P.

(b) In this part question the candidates were required to find the value of *T*. As the equation obtained in **part (a)** had two unknowns (*T* and *u*), a second equation was needed, that is the equation involving the vertical displacement of the two particles. Some candidates had no problems in writing the second equation and then obtaining the correct value for *T*, in many cases with clear explanations and diagrams. Those candidates who used the equation of the vertical displacements in **part (a)** usually took the same approach in **part (b)** which was incorrect.

(c) Most of the candidates who answered **part (b)** correctly had no problems reaching a fully correct solution in this part. The best answers included both the value -24 (metres) for the vertical displacement and an indication that this was 24 metres below point *O*, including a diagram.

FURTHER MATHEMATICS

Paper 9231/32
Further Mechanics

Key messages

A diagram is often an invaluable tool in helping a candidate to make good progress. This is particularly the case when forces or velocities are involved. If a diagram is given on the question paper it may be sufficient to annotate that diagram, although candidates are always free to draw their own diagrams as well.

When a result is given in a question, candidates must take care to give sufficient detail in their working, so that their solution is clear and complete. In all questions, however, candidates are advised to show all their working, as method is important as well as accuracy.

General comments

Candidates are encouraged to draw a suitable diagram as this helps them to understand the problem and model it correctly. For example, in **Question 3**, candidates who drew a clear diagram realised that the acceleration of the particle had a negative sign and were able to write in the correct form the differential equation describing its motion.

Candidates should pay particular attention to the mathematical vocabulary used in the text of the question and also to what the question asks them to do. For example, **Question 4** used the word *shell* three times and mentioned a *hollow cylinder* twice, and **Question 6(a)** asked for the speed of sphere A in addition to that of sphere B. Candidates who understood the text correctly managed to set up the correct moments equation in **Question 4** and to obtain a fully correct solution in **Question 6(a)**.

Candidates are reminded that, when the answer is given, they are expected to show their working in full, even if it involves the use of simple algebra. This was relevant to **Question 6(a)** and **Question 7(a)**.

Comments on specific questions

Question 1

(a) Most candidates had no difficulty in stating the values of the components of the tension in the string and then went on to answer the question correctly. Better responses used Pythagorean triples, whereas weaker responses determined the magnitude of the angle θ, as an intermediate step, even though this was not needed. A common mistake was to consider the vertical component of the tension as 10 g, i.e.,100 N, rather than 10 N as stated in the question.

(b) This part of the question was also well answered by many candidates. They needed to apply Hooke's law correctly to determine the extension of the string and then obtain the answer using trigonometry. A common mistake was to confuse sine and cosine thus obtaining an answer of 0.6, or to omit to calculate the extension.

Question 2

This question proved challenging for some candidates, who had difficulty in writing the correct equation for the conservation of energy, often making errors in the signs of the different terms. Some candidates worked out the value of the tension of the string at point A, even though this was not necessary. A common error was to state that the velocity of the particle is zero when the string goes slack, and to use that value in the equation for the conservation of energy. In the best responses, candidates drew very clear diagrams and used them effectively to obtain the correct answer, often in an elegant and efficient way.

Question 3

Many candidates were able to set up a differential equation and to solve it, including applying the boundary conditions. However, only some candidates realised that acceleration and velocity had opposite directions and therefore the correct differential equation was $\frac{\mathrm{d}v}{\mathrm{d}t} = -\frac{4000}{(5t+4)^3}$. Candidates who drew a diagram representing the situation and included the directions of acceleration and velocity were able to write the correct differential equation. They usually went on to provide a correct solution.

Question 4

(a) This part of the question was answered correctly by few candidates. Most candidates wrote the equation of a hemisphere on top of a solid cylinder, even though the question explicitly mentioned a hemispherical shell and a hollow cylinder. Weaker responses included a moments equation where the terms were not consistent dimensionally. This is an easy check to perform, and candidates are encouraged to use it. Without a dimensionally consistent equation it is not possible to proceed further.

(b) In better responses, candidates drew a correct diagram to answer this part question. With the aid of the diagram, they were able to obtain the inequality linking $\tan\theta$, a, h and $\bar{x}$ (the distance of the centre of the mass obtained in the previous part) and to solve it. Common errors involved writing an equation instead of an inequality, writing the incorrect direction of the inequality, or comparing $\tan\theta$ with $\frac{\bar{x}}{a}$ and not with $\frac{a}{\bar{x}}$.

Question 5

To obtain the correct answer for this question, candidates were required to perform three steps:

- Apply Newton's Second Law horizontally at point *A*. They needed to use trigonometry to determine the expression for the radius and simplify the equation to determine the tension on the string at point *A*: $T = 3am\omega^2$.
- Apply Newton's Second Law and proceed to find the tension at point *B*: $T = \frac{3}{4}amk^2\omega^2$.
- Equate the expressions for the tensions and solve the resulting equation in k to give the solution $k = 2$.

Some excellent responses were seen as the question gave candidates the opportunity to respond in an elegant and efficient way.

Some candidates performed the first two steps very quickly, but then did not realise that the tension at point *A* was equal to that at point *B* as those were the endpoint of the string. Consequently, they rarely managed to obtain the correct answer.

Other candidates found this question challenging in various ways. Some did not apply Newton's Second Law correctly or could not determine the correct expression for the radii of the two circles. A few candidates did not use the correct value for the mass of the particle attached at point *B*.

Question 6

(a) Many candidates showed a great degree of confidence in writing and solving successfully the system of equations obtained from the conservation of linear momentum and Newton's experimental law to obtain the given answer. However, some of them omitted to determine the expression for the speed of sphere A along the line of centres, even though this was explicitly requested in the question. Those candidates did not produce a fully correct solution in this part although, in some cases, they worked it out subsequently in **part (b)**. Even though it was not required in **part (a)**, some candidates worked out the expression for the speed of sphere A, including the component perpendicular to the line of centres. They later used it in **part (b)**, usually with good success. A common mistake was not to include the mass in the equation for the conservation of momentum, i.e., writing $v + kw = u\cos\alpha$ instead of $mv + kmw = mu\cos\alpha$. As a result, these candidates did not provide a complete argument leading to the given answer as required.

(b) This part question proved more challenging than **part (a)**. Many candidates had difficulty in determining the kinetic energy of sphere *A* after the collision. The most common mistake was to omit the component of the speed perpendicular to the line of centres $(u\sin\alpha)$. Another common mistake was to write the mass of sphere *B* as *m*, rather than *km*. Those who wrote a correct equation for the kinetic energy of the spheres usually had no difficulty in solving the equation to obtain the correct answer using competent algebraic manipulation.

Question 7

(a) This part question required candidates to obtain a given equation by working out expressions for the <u>horizontal</u> displacements of particles P and Q. Candidates who wrote the correct displacements had no problem in obtaining the required answer, often in a convincing and elegant manner. Weaker responses considered the <u>vertical</u> displacements of the particles instead which was incorrect. Another common mistake was to use time $T+1$ for particle Q and T for particle P.

(b) In this part question the candidates were required to find the value of *T*. As the equation obtained in **part (a)** had two unknowns (*T* and *u*), a second equation was needed, that is the equation involving the vertical displacement of the two particles. Some candidates had no problems in writing the second equation and then obtaining the correct value for *T*, in many cases with clear explanations and diagrams. Those candidates who used the equation of the vertical displacements in **part (a)** usually took the same approach in **part (b)** which was incorrect.

(c) Most of the candidates who answered **part (b)** correctly had no problems reaching a fully correct solution in this part. The best answers included both the value -24 (metres) for the vertical displacement and an indication that this was 24 metres below point *O*, including a diagram.

FURTHER MATHEMATICS

Paper 9231/33
Further Mechanics

Key messages

When a result is given in a question, candidates must take care to give sufficient detail in their working so that their solution is clear and complete. In all questions, however, candidates are advised to show all their working, as method is important as well as accuracy.

A diagram is often an invaluable tool in helping a candidate to make good progress. This is particularly the case when forces or velocities are involved. If a diagram is given on the question paper it may be sufficient to annotate that diagram, although candidates are always free to draw their own diagrams as well.

General comments

In most questions the majority of candidates understood which method to use. Some candidates omitted to draw a suitable diagram, or to annotate the given diagram, and this resulted in incorrect equations. This was particularly the case in **Question 1** and **Question 4**.

Question 7 is not intrinsically difficult, but it proved particularly challenging for candidates. Many candidates incorrectly used resolution equations on the part of the system that was in equilibrium. Please see below for further details on this.

Candidates are reminded that, when the answer is given, they are expected to show their working in full, even if it involves the use of simple algebra. This was relevant to **Question 5(a)**.

Comments on specific questions

Question 1

This question was answered correctly by the majority of candidates. For the moments equation, the lamina was usually split into a rectangle and two triangles. Some candidates preferred to subtract two triangles from a rectangle with vertices at *O*, *A*, (15, 4) and (0, 4). Other candidates extended *OC* and *AB* to meet at the point *D* and then used triangles *OAD* and *CBD*. Whichever method was used, the most common error was in determining the position of the centre of mass of a triangle. Candidates knew that the ratio 1: 2 was involved but were often confused in applying this.

Question 2

The majority of candidates successfully found the loss in kinetic energy, the gain in elastic potential energy and the work done against the frictional force and combined them in a correct energy/work done equation. The most common error was a sign error involving the work done term. A minority of candidates omitted a work done term.

Question 3

(a) Most candidates began by writing down the horizontal and vertical components of the velocity of *P* at time *t*. They then squared and added their two expressions and equated the result to the square of the given speed $25\,\text{m}\,\text{s}^{-1}$. The resulting equation was almost always simplified accurately leading to the correct final answer.

(b) The most popular approach in this part was to find the time taken to the greatest height, double this to give the time of flight and use the time to find the horizontal range. Candidates who simply quoted memorised formulae often confused the time to the greatest height with the time of flight. A minority of candidates used the equation of the trajectory together with the condition $y = 0$ to find the required range successfully.

Question 4

(a) Some very elegant and concise solutions to this question were seen. Drawing a diagram labelled with all the given information was an essential first step to ensuring accuracy in solving this problem. Those candidates who attempted to work without a diagram often had sign errors and/or the defined angles incorrectly placed.

To solve this problem, it was necessary to use the energy equation for the motion of the particle from its initial position at *A* to its final position when the string makes an angle α with the downward vertical through *O*. The majority of candidates did this accurately. The tensions in the string at the initial and final positions of the particle are found by applying Newton's Second at each position. Finally, one of the unknown tensions can be eliminated using the given information that the magnitude of the final tension is equal to 10 times the magnitude of the tension at *A*.

Most candidates followed this procedure and many candidates obtained the correct expression for the tension at *A*. Some candidates had sign errors in their equations but proceeded to find a value for the tension. A minority of candidates obtained the correct equations but were unable to find a way to eliminate unknowns in order to isolate the tension.

(b) This part was answered correctly by almost all candidates who had successfully navigated **part (a)**.

Question 5

(a) Many candidates were able to set up a differential equation and to solve it, including applying the boundary conditions. However, there were many sign errors that disappeared so that, when the given result was stated, the solution was not fully correct. The main cause of a sign error was in setting up the differential equation. The force was given as a resistive force and so there was a minus sign on the right hand side when Newton's Second Law was applied:
$4v\dfrac{dv}{dx} = -\left(4e^{-x} + 12\right)e^{-x}$. Other candidates made the error of assuming that the resistive force was equal to the acceleration, disregarding the mass. Solutions to problems of this type should always start by writing down Newton's Second Law.

The majority of candidates worked accurately throughout and obtained the given result for v convincingly. A significant minority of candidates obtained an incorrect expression for v^2 then wrote down the given result so their argument was not complete.

(b) The integration required in this part proved to be a stumbling block for a significant minority of candidates. They should recognise $\int \dfrac{e^x}{3e^x + 1}dx$ as a case where the numerator of the integrand is a constant multiple of the differential of the denominator, and it therefore integrates to $\dfrac{1}{3}\ln\left(3e^x + 1\right)$. Those candidates who realised this were usually able to proceed to find a correct expression for x in terms of t. A few candidates stopped before the final step, giving $e^x = \dfrac{4}{3}e^{3t} - \dfrac{1}{3}$ as their answer.

Question 6

Many candidates found this question quite challenging, particularly **part (b)**.

(a) A significant number of candidates appeared to be unfamiliar with a collision between a particle and a wall instead of the more usual collision between two spheres. It was necessary to use conservation of linear momentum parallel to the wall and Newton's law of restitution perpendicular to the wall. These led to two simple equations involving the velocity before and after the collision and the angle between the direction of travel before and after the collision. The square of the final velocity could then be written as $(u\cos\theta)^2 + (eu\sin\theta)^2$. Combined with the given information about the kinetic energies before and after the collision, this could be used to find the value of *e*.

(b) Only some candidates were able to make any progress in this part. The first step was to determine the value of α, the angle between the velocity of the particle after the first collision and the first wall. This could be found by dividing the momentum and restitution equations found in **part (a)** to obtain $\tan\alpha = e\tan\theta$. With the value of *e* found in **part (a)** and the given value of θ, it is easily shown that $\alpha = 45°$. Using this together with the geometry of the situation for the collision with the second wall, it follows that $\tan\beta = e\tan(120 - \alpha)$, and so $\tan\beta = 61.8°$.

Question 7

(a) The majority of candidates annotated the given diagram with all the relevant forces at *D* and *E*, in the relevant directions: normal reactions, frictional forces and weights. They then had to make choices about resolving forces and forming moments equations. The most efficient method is to take moments about *D* and about *E*. From these two equations, the relationship between *N* and *R* can be found. Of course, moments can be taken about any point, although the resulting equations will not all be quite as simple. Another approach is to take moments about one point and then also resolve in two directions for the whole system.

A very common error was to resolve perpendicular to each rod, individually, giving the equations $R = W\cos\theta$, $N = W\sin\theta$. These results are not correct as they are based on a misunderstanding that parts of the system can be considered in isolation. This is not so since all resolving of forces must include all the forces acting on the whole rigid body. The majority of candidates adopted this invalid approach and as a result very few fully correct solutions were seen.

(b) Very few candidates answered this part successfully.

FURTHER MATHEMATICS

Paper 9231/41
Further Probability & Statistics

Key messages

In all questions candidates are advised to show all their working as method is important as well as accuracy. When a result is given in a question, candidates must give sufficient detail in their working so that their solution is clear and complete.

Care must be taken with the language used when interpreting the result of any test. In general, a hypothesis test is not a proof and it is not appropriate to use definitive statements. Concluding statements should always include some degree of uncertainty, for example, 'there is insufficient evidence to support the claim that ...' rather than 'the test proves that ...'.

General comments

Almost all candidates attempted all the questions. The standard was generally high, with many candidates presenting clear and accurate solutions throughout.

The rubric for this paper specifies that non-exact numerical answers should be given to 3 significant figures. Candidates are therefore advised to use a greater degree of accuracy while working towards the final answer. Premature rounding or working to only 3 significant figures throughout may result in an error in the final answer. This is particularly the case in statistics problems where several different values are calculated, each depending on the previous one. Such rounding errors were often seen in **Question 4(b)**.

Comments on specific questions

Question 1

(a) The majority of candidates used a paired t-test and made good progress in reaching a fully correct solution. The most common errors were arithmetical errors in calculating the differences. Some candidates worked with 'before' minus 'after' both in the hypotheses and in the final conclusion. This usually led to confusion in interpreting the result. A few candidates did not work with differences, an incorrect approach.

(b) Only a minority of candidates gave a satisfactory answer to this part. The assumption is that the distribution of the population differences is normal. Most candidates gave one or two of these requirements, with the generic answer 'distribution is normal' being the most common.

Question 2

(a) This part was almost always answered correctly, with only the occasional calculation error.

(b) This part was found to be challenging for many candidates. Most candidates realised that they needed to find an expression which, when squared, would give the probability generating function of Y, but then could not find such an expression. Some candidates simply took the square root of each term in the given quartic expression. The best responses found a quadratic expression which, when squared, was equal to the given quartic expression: $\mathrm{G}_Y(t) = \left(a + bt + ct^2\right)^2$. A simple comparison of coefficients with the given quartic enabled a, b and c to be found. An alternative approach is to factorise the given probability generating function of Y as $\left(\frac{1}{10}(1+t)(2+3t)\right)^2$.

Question 3

(a) This part was answered well, although a small number of candidates did not show sufficient working to justify the given answer.

(b) Most candidates found the value of the variance accurately. A few omitted the subtraction of the 'mean squared' from their calculation.

(c) This part was less well answered. The main cause of error was omitting to find in which region of the piecewise probability density function the median must lie. This can be determined by evaluating the integral $\int_0^2 kx(4-x)dx$ as 0.4. Since 0.4 is less than 0.5, the median must lie between 2 and 6. The value of the median, m, can be found by solving the equation $\int_2^m k(6-x)\mathrm{d}x = 0.5 - 0.4$ or alternatively $\int_m^6 k(6-x)\mathrm{d}x = 0.5$. The most common error was to omit the 0.4 and simply work with $\int_2^m k(6-x)\mathrm{d}x = 0.5$.

Question 4

(a) This part was generally answered well by candidates, although the instruction to give answers correct to 2 decimal places was not always followed.

(b) Almost all candidates recognised the need for a chi-squared test, but few provided fully correct solutions. Since the value of q was found to be 1.70 in **part (a)**, the expected frequencies in the final two columns sum to 5.04. These two columns needed to be combined before calculating the contributions to the test statistic. Many candidates added three columns together, and some candidates did not combine columns at all. The hypotheses were often stated with insufficient detail, for example 'it is a good fit model' or 'the distribution is a good fit'. It is expected that both the distribution and the data are mentioned in the hypotheses. Examples are 'the given Poisson distribution is a good model for the data' or, as a minimum, 'the distribution Po(2.5) fits the data'. A minority of candidates did not include any level of uncertainty in their conclusion.

Some candidates did not show any working when finding the test statistic, presumably using a calculator for the whole process. This is a risky strategy: candidates should realise that a response with no working is only acceptable if the value of the test statistic is correct.

Question 5

This question was answered well by most candidates.

(a) Many completely correct solutions were seen. Some candidates made arithmetical errors, some confused standard deviation and variance, and some used an incorrect value for t in the formula for the confidence interval. A small number of candidates used a normal distribution instead of the t-distribution required for a small sample.

(b) The errors seen in this part were usually in formulating the hypotheses and writing the conclusion. The conclusion must be given in context in words, not symbols, and there must be a level of uncertainty in the language used.

Question 6

This question required an application of the Wilcoxon signed-rank test. Because of the size of the sample, a normal approximation was required. Many candidates used the correct approach to find the P and Q values from the data, but there were many accuracy errors. Each data value is subtracted from the median 18, and these signed differences are ranked, with signs. Candidates who drew up a table with all the relevant values were usually more successful in working accurately than those who annotated the data in the question paper. Either approach is acceptable but, in this test, accuracy is of paramount importance.

Only a minority of candidates stated suitable hypotheses. Some candidates used μ instead of m, and others did not use the given median of 18 but opted instead for 0.

The majority of candidates found the mean and variance of the normal distribution correctly though not all of them made further progress in finding the signed ranks. The test statistic was usually found correctly, but a common error was to omit the continuity correction. Other candidates attempted to use a one-tail test, comparing with –1.282 instead of –1.645, and this was often inconsistent with their stated hypotheses. Candidates who opted to make an area comparison needed to compare their value with 0.1 instead of 0.05.

FURTHER MATHEMATICS

Paper 9231/42
Further Probability & Statistics

Key messages

In all questions candidates are advised to show all their working as method is important as well as accuracy. When a result is given in a question, candidates must give sufficient detail in their working so that their solution is clear and complete.

Care must be taken with the language used when interpreting the result of any test. In general, a hypothesis test is not a proof and it is not appropriate to use definitive statements. Concluding statements should always include some degree of uncertainty, for example, 'there is insufficient evidence to support the claim that ...' rather than 'the test proves that ...'.

General comments

Almost all candidates attempted all the questions. The standard was generally high, with many candidates presenting clear and accurate solutions throughout.

The rubric for this paper specifies that non-exact numerical answers should be given to 3 significant figures. Candidates are therefore advised to use a greater degree of accuracy while working towards the final answer. Premature rounding or working to only 3 significant figures throughout may result in an error in the final answer. This is particularly the case in statistics problems where several different values are calculated, each depending on the previous one. Such rounding errors were often seen in **Question 4(b)**.

Comments on specific questions

Question 1

(a) The majority of candidates used a paired *t*-test and made good progress in reaching a fully correct solution. The most common errors were arithmetical errors in calculating the differences. Some candidates worked with 'before' minus 'after' both in the hypotheses and in the final conclusion. This usually led to confusion in interpreting the result. A few candidates did not work with differences, an incorrect approach.

(b) Only a minority of candidates gave a satisfactory answer to this part. The assumption is that the distribution of the population differences is normal. Most candidates gave one or two of these requirements, with the generic answer 'distribution is normal' being the most common.

Question 2

(a) This part was almost always answered correctly, with only the occasional calculation error.

(b) This part was found to be challenging for many candidates. Most candidates realised that they needed to find an expression which, when squared, would give the probability generating function of Y, but then could not find such an expression. Some candidates simply took the square root of each term in the given quartic expression. The best responses found a quadratic expression which, when squared, was equal to the given quartic expression: $G_Y(t) = \left(a + bt + ct^2\right)^2$. A simple comparison of coefficients with the given quartic enabled a, b and c to be found. An alternative approach is to factorise the given probability generating function of Y as $\left(\frac{1}{10}(1+t)(2+3t)\right)^2$.

Question 3

(a) This part was answered well, although a small number of candidates did not show sufficient working to justify the given answer.

(b) Most candidates found the value of the variance accurately. A few omitted the subtraction of the 'mean squared' from their calculation.

(c) This part was less well answered. The main cause of error was omitting to find in which region of the piecewise probability density function the median must lie. This can be determined by evaluating the integral $\int_0^2 kx(4-x)\,dx$ as 0.4. Since 0.4 is less than 0.5, the median must lie between 2 and 6. The value of the median, m, can be found by solving the equation $\int_2^m k(6-x)\,\mathrm{d}x = 0.5 - 0.4$ or alternatively $\int_m^6 k(6-x)\,\mathrm{d}x = 0.5$. The most common error was to omit the 0.4 and simply work with $\int_2^m k(6-x)\,\mathrm{d}x = 0.5$.

Question 4

(a) This part was generally answered well by candidates, although the instruction to give answers correct to 2 decimal places was not always followed.

(b) Almost all candidates recognised the need for a chi-squared test, but few provided fully correct solutions. Since the value of q was found to be 1.70 in **part (a)**, the expected frequencies in the final two columns sum to 5.04. These two columns needed to be combined before calculating the contributions to the test statistic. Many candidates added three columns together, and some candidates did not combine columns at all. The hypotheses were often stated with insufficient detail, for example 'it is a good fit model' or 'the distribution is a good fit'. It is expected that both the distribution and the data are mentioned in the hypotheses. Examples are 'the given Poisson distribution is a good model for the data' or, as a minimum, 'the distribution Po(2.5) fits the data'. A minority of candidates did not include any level of uncertainty in their conclusion.

Some candidates did not show any working when finding the test statistic, presumably using a calculator for the whole process. This is a risky strategy: candidates should realise that a response with no working is only acceptable if the value of the test statistic is correct.

Question 5

This question was answered well by most candidates.

(a) Many completely correct solutions were seen. Some candidates made arithmetical errors, some confused standard deviation and variance, and some used an incorrect value for t in the formula for the confidence interval. A small number of candidates used a normal distribution instead of the t-distribution required for a small sample.

(b) The errors seen in this part were usually in formulating the hypotheses and writing the conclusion. The conclusion must be given in context in words, not symbols, and there must be a level of uncertainty in the language used.

Question 6

This question required an application of the Wilcoxon signed-rank test. Because of the size of the sample, a normal approximation was required. Many candidates used the correct approach to find the P and Q values from the data, but there were many accuracy errors. Each data value is subtracted from the median 18, and these signed differences are ranked, with signs. Candidates who drew up a table with all the relevant values were usually more successful in working accurately than those who annotated the data in the question paper. Either approach is acceptable but, in this test, accuracy is of paramount importance.

Only a minority of candidates stated suitable hypotheses. Some candidates used μ instead of m, and others did not use the given median of 18 but opted instead for 0.

The majority of candidates found the mean and variance of the normal distribution correctly though not all of them made further progress in finding the signed ranks. The test statistic was usually found correctly, but a common error was to omit the continuity correction. Other candidates attempted to use a one-tail test, comparing with –1.282 instead of –1.645, and this was often inconsistent with their stated hypotheses. Candidates who opted to make an area comparison needed to compare their value with 0.1 instead of 0.05.

FURTHER MATHEMATICS

Paper 9231/43
Further Probability & Statistics

Key messages

In all questions candidates are advised to show all their working as method is important as well as accuracy. When a result is given in a question, candidates must give sufficient detail in their working so that their solution is clear and complete.

Care must be taken with the language used when interpreting the result of any test. In general, a hypothesis test is not a proof and it is not appropriate to use definitive statements. Concluding statements should always include some degree of uncertainty, for example, 'there is insufficient evidence to support the claim that ...' rather than 'the test proves that ...'.

General comments

Almost all candidates attempted all the questions. The standard was generally high, with many candidates presenting clear and accurate solutions throughout.

The rubric for this paper specifies that non-exact numerical answers should be given to 3 significant figures. Candidates are therefore advised to use a greater degree of accuracy while working towards the final answer. Rounding prematurely, for example working to only 3 significant figures, may result in inaccuracy in the final answer. This is particularly the case in statistics problems where several different values are calculated, each depending on the previous one. Such rounding errors were often seen in **Question 6**.

Comments on specific questions

Question 1

This question was answered well by the majority of candidates. The most common error was an incorrect t value used in the confidence interval formula. A small minority of candidates opted for a z value, which is not appropriate for a small sample with unknown population variance.

Question 2

Almost all candidates recognised the need for a chi-squared test and carried out the test, but relatively few solutions were fully correct. The hypotheses were often stated with insufficient detail. It was necessary to mention 'size of shell', 'independent' and 'beach location' or simply to copy the words in the question: 'size of shell is independent of beach location'. A few candidates reduced the hypotheses to 'independent' and 'not independent' which did not refer to the context. Candidates are reminded to include the context of a question in both their hypotheses and their conclusion.

The conclusion also needs to be written with a level of uncertainty in the words used. It was not unusual to see 'the evidence proves that the shell size is dependent on the beach location'. A hypothesis test cannot be used to prove anything, only to provide some evidence to suggest an association. A statement beginning 'there is insufficient evidence to suggest that …' is more appropriate.

Question 3

(a) Most candidates were able to make the connection between the probabilities relating to the two coins and the given coefficients of the probability generating function. Consequently, they were able to form two equations in *a* and *b*. A minority of candidates were not able to make progress in solving these equations to find the value of *a*.

(b) From the information given in the stem of the question, it is possible to deduce that $G_X(t)$ is equal to $\frac{6}{12}+\frac{5}{12}t+\frac{1}{12}t^2$. Squaring this quadratic expression gives $G_Y(t)$. A significant number of candidates assumed $G_X(t) = \frac{5}{12}t+\frac{1}{12}t^2$. A quick check reveals that this cannot be an expression for a probability generating function. The coefficients of the powers of *t* must sum to one. Furthermore, in practical terms, the missing term is the probability that no heads are obtained when the two coins are thrown. This is clearly not zero. Candidates are advised to check that the coefficients of any probability generating function they are working with sum to one.

(c) Almost all responses showed correct use of methods, but only those candidates who worked accurately throughout the question could reach a correct answer.

Question 4

The majority of candidates answered this question well.

(a) The most common error that occurred in this part was in misunderstanding the meaning of $E(\sqrt{X})$. Instead of the correct expression $\int \sqrt{x}\,\frac{3}{8}\left(1+\frac{1}{x^2}\right)dx$, a minority of candidates used either $\int \sqrt{x}\,\frac{3}{8}\left(1+\frac{1}{x}\right)dx$ or $\int x\,\frac{3}{8}\left(1+\frac{1}{x}\right)dx$.

(b) Most candidates showed a good understanding of the relationship between a probability density function and a cumulative density function. They were able to carry out the change of variable accurately though some errors occurred in the integration.

(c) Most candidates used a cumulative density function to form a quadratic equation which they then solved. A minority of candidates were confused about whether they were working with *X* or *Y* and gave a final answer of $\frac{5}{3}$ instead of $\frac{25}{9}$.

Question 5

This question required candidates to apply both the Wilcoxon signed-rank test and the sign test to a set of data. Candidates were very familiar with the first test and the vast majority were able to carry out the test correctly. However, a substantial number of candidates appeared unfamiliar with the basic methodology required to carry out a sign test and could make little or no progress with it.

The first step in applying the sign test is to find the sign of the difference between each piece of data and the stated median 5.50. In this case, there are 8 positive differences (and 3 negative differences). The binomial distribution B(11, 0.5) is used to find $P(X \geqslant 8)$ or $P(X \leqslant 3)$. The result, 0.113, is then compared with 0.05 since this is a two-tail test. Alternatively, 0.227 can be compared with 0.1. The small number of candidates who understood how to apply the sign test did so successfully.

Question 6

(a) This is a standard question on testing the difference between the population means of two large samples using a *z*-test. The majority of candidates provided solutions that were fully correct or almost correct.

A minority of candidates assumed that the two distributions shared a common population variance and so used a pooled estimate. This was incorrect because the wording of the question did not suggest that the assumption might be valid.

Some candidates gave insufficient working to support their calculations. While intermediate answers can be implied by correct subsequent values, candidates are recommended to give the intermediate values. In this situation, a response with no working is only acceptable if the solution is correct. Since the accuracy required for final answers is 3 significant figures, it is essential that candidates work either exactly (with fractions) or to at least 4 significant figures prior to rounding for the final answer.

(b) The question asks for the possible significance levels for which a difference of means is greater than 0.25. It is necessary to consider the expression $\frac{0.6-0.25}{s}$, where s is the variance found in **part (a)**. Only a few candidates were able to make this first step and most of these successfully reached a correct answer.